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LETTER TO THE EDITOR

Spread of damage in the discrete N-vector ferromagnet: exact results

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Abstract. We establish exact relations between damage spreading and relevant thermal quantities (appropriate order parameters and correlation functions) of the ferromagnetic discrete N-vector model on an arbitrary lattice and for arbitrary ergodic dynamics. These relations recover, as particular cases, those already existing in the literature for the Ising, Potts and Ashkin–Teller models.

Since the recent and interesting Coniglio *et al* [1] connection between spread of damage and relevant thermal equilibrium quantities, some effort has been dedicated to the discussion, from this new standpoint, of discrete statistical models. In fact, the Coniglio *et al* study of the Ising ferromagnet has been recently [2] extended to the q-state Potts and Ashkin–Teller ferromagnets. It is important to remark that, although this type of approach is close to the now standard calculations of the Hamming distance (see, for instance, [3–6]), the present relations are based on *specific* combinations of damages. These combinations are, in general, essentially different from the Hamming distance (with the exception of the Ising ferromagnet with heat-bath dynamics, if a special initial condition is chosen [1]).

We consider the following dimensionless Hamiltonian (discrete N-vector or cubic model)

$$\beta\mathscr{H} = -N\sum_{ij}\{KS_i\cdot S_j + NL(S_i\cdot S_j)^2\} \tag{1}$$

where $\beta \equiv 1/k_BT$, (i, j) run over all the pairs of connected sites on an arbitrary lattice, $K>0$ and $K+NL>0$ (in order to guarantee a *ferromagnetic* fundamental state) and S_i is an N-component vector which points along the edges of an N-dimensional hypercube, i.e. $S_i=(\pm1, 0, 0, \ldots, 0)$. $(0, \pm1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, \pm1)$. For theoretical and experimental work related to this Hamiltonian see [7] and references therein. Hamiltonian (1) can be conveniently rewritten as follows. We introduce, for each site, a new random variable α_i such that $\alpha_i=1$ $(\alpha_i=-1)$ whenever $S_i=(1, 0, 0, \ldots, 0)$ $(S_i=(-1, 0, 0, \ldots, 0))$, $\alpha_i=2$ $(\alpha_i=-2)$ whenever $S_i=(0, 1, 0, \ldots, 0)$

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($S_i=(0, -1, 0, \ldots, 0)$), $\ldots$, and $\alpha_i=N$ ($\alpha_i=-N$) whenever $S_i=(0, 0, 0, \ldots, 1)$ ($S_i=0, 0, 0, \ldots, -1)$). The Hamiltonian becomes now

$$\beta\mathscr{H}=-N\sum_{ij}\{(K+NL)\delta(\alpha_i, \alpha_j)+(-K+NL)\delta(\alpha_i, -\alpha_j)\} \tag{2}$$

where $\delta(\alpha_i, \alpha_j)$ denotes Kroenecker's delta function. At high temperatures, this model is *paramagnetic* (P). At low temperatures the system might exhibit long range ordering (this typically occurs for lattice Euclidian or fractal dimension $d>1$). If this is the case, either it goes (for decreasing temperature) from the paramagnetic to a *ferromagnetic* (F) phase, or first goes to an *intermediate* (I) phase, and then to the ferromagnetic one ([7] and references therein). In the paramagnetic phase, all N axes and both senses on each of them are equally probable. In the intermediate phase, one among the N axes is preferently occupied (say $\alpha_i^2=1, \forall i$), but both senses of this axis are equally probable. Finally, in the ferromagnetic phase, one of those two senses (say $\alpha_i=1$, $\forall$i) is preferently occupied. The associated order parameters can be conveniently defined as follows:

$$m_I\equiv\langle\delta(\alpha_i^2, 1)\rangle-\frac{1}{N} \tag{3}$$

and

$$m_F\equiv\langle\delta(\alpha_i, 1)\rangle-\langle\delta(\alpha_i, -1)\rangle \tag{4}$$

where $\langle\ldots\rangle$ denotes the thermal canonical average. In the P-phase we have $m_I=m_F=0$; in the I-phase we have $m_I\neq 0$ and $m_F=0$; finally, in the F-phase we have $m_I\neq 0$ and $m_F\neq 0$.

Let us now introduce the following convenient two-body correlation functions:

$$\Gamma_I(i,j)\equiv\langle\delta(\alpha_i^2, 1)\delta(\alpha_j^2, 1)\rangle-\langle\delta(\alpha_i^2, 1)\rangle\langle\delta(\alpha_j^2, 1)\rangle \tag{5}$$

and

$$\Gamma_F(i,j)\equiv\langle\delta(\alpha_i, 1)\delta(\alpha_j, 1)\rangle-\langle\delta(\alpha_i, 1)\rangle\langle\delta(\alpha_j, 1)\rangle. \tag{6}$$

It is now appropriate to specify *four* different types of local damages between two copies (hereafter referred to as A and B) of the system. These two copies are assumed to evolve in time under one and the same *ergodic* dynamics (Metropolis, heat-bath, Glauber [8], generalized [9] or any other one satisfying detailed balance) with the *same* sequence of random numbers for updating corresponding sites of copies A and B. The four local damages we shall consider are

$$\begin{aligned}
&\alpha_i^A=1 \text{ and } \alpha_i^B\neq 1\\
&\alpha_i^A\neq 1 \text{ and } \alpha_i^B=1\\
&(\alpha_i^A)^2=1 \text{ and } (\alpha_i^B)^2\neq 1\\
&(\alpha_i^A)^2\neq 1 \text{ and } (\alpha_i^B)^2=1.
\end{aligned} \tag{7}$$

The four corresponding occurrence probabilities are given by

$$\begin{aligned}
&p_1\equiv[\delta(\alpha_i^A, 1)(1-\delta(\alpha_i^B, 1))]\\
&p_2\equiv[(1-\delta(\alpha_i^A, 1))\delta(\alpha_i^B, 1)]\\
&p_3\equiv[\delta((\alpha_i^A)^2, 1)(1-\delta((\alpha_i^B)^2, 1))]\\
&p_4\equiv[(1-\delta((\alpha_i^A)^2, 1))\delta((\alpha_i^B)^2, 1)]
\end{aligned} \tag{8}$$

where [. . .] denotes the *time* average over the trajectory in phase space; in other words, we are interested in the frequencies of occurrence, along time, of those particular damages at site *i*. Let us now define

$$F_1 \equiv p_1 - p_2 = [\delta(\alpha_i^A, 1)] - [\delta(\alpha_i^B, 1)] \tag{9}$$

and

$$F_2 \equiv p_3 - p_4 = [\delta((\alpha_i^A)^2, 1)] - [\delta((\alpha_i^B)^2, 1)]. \tag{10}$$

We now need to introduce four different *constrained* (time) evolutions:

(i) Copy A evolves without any constraint; copy B evolves by imposing, at all times and for an arbitrarily chosen site (say $i=0$), $\alpha_0^B \neq 1$.

(ii) Copy A evolves with $\alpha_0^A = 1$; copy B evolves with $\alpha_0^B \neq 1$.

(iii) Copy A evolves without any constraint; copy B evolves with $(\alpha_0^B)^2 \neq 1$.

(iv) Copy A evolves with $(\alpha_0^A)^2 = 1$; copy B evolves with $(\alpha_0^B)^2 \neq 1$.

Evolution (i) implies that

$$[\delta(\alpha_i^A, 1)] = \langle \delta(\alpha_i, 1) \rangle \tag{11}$$

and (by using conditional probability)

$$[\delta(\alpha_i^B, 1)] = \frac{\langle \delta(\alpha_i, 1)(1 - \delta(\alpha_0, 1)) \rangle}{\langle 1 - \delta(\alpha_0, 1) \rangle} \tag{12}$$

where we have used ergodicity. By substituting (11) and (12) into (9) we straightforwardly obtain

$$F_{11} \equiv F_1(\text{evolution(i)}) = \frac{\Gamma_F(i, 0)}{1 - \langle \delta(\alpha_0, 1) \rangle}. \tag{13}$$

Evolution (ii) implies that

$$[\delta(\alpha_i^A, 1)] = \frac{\langle \delta(\alpha_i, 1) \delta(\alpha_0, 1) \rangle}{\langle \delta(\alpha_0, 1) \rangle}. \tag{14}$$

Since copy B evolves here as it did in evolution (i), equation (12) still holds. By substituting (12) and (14) into (9) we obtain

$$F_{12} \equiv F_1(\text{evolution(ii)}) = \frac{\Gamma_F(i, 0)}{\langle \delta(\alpha_0, 1) \rangle (1 - \langle \delta(\alpha_0, 1) \rangle)}. \tag{15}$$

Evolution (iii) implies that

$$[\delta((\alpha_i^A)^2, 1)] = \langle \delta(\alpha_i^2, 1) \rangle \tag{16}$$

as well as

$$[\delta((\alpha_i^B)^2, 1)] = \frac{\langle \delta(\alpha_i^2, 1)(1 - \delta(\alpha_0^2, 1)) \rangle}{\langle 1 - \delta(\alpha_0^2, 1) \rangle}. \tag{17}$$

By substituting (16) and (17) into (10) we obtain

$$F_{21} \equiv F_2(\text{evolution(iii)}) = \frac{\Gamma_I(i, 0)}{1 - \langle \delta(\alpha_0^2, 1) \rangle}. \tag{18}$$

Finally, evolution (iv) implies that

$$[\delta((\alpha_i^A)^2, 1)] = \frac{\langle \delta(\alpha_i^2, 1)\delta(\alpha_0^2, 1)\rangle}{\langle \delta(\alpha_0^2, 1)\rangle}. \tag{19}$$

Equations (17) (which still hold) and (19) placed into (10) yield

$$F_{22} \equiv F_2(\text{evolution(iv)}) = \frac{\Gamma_I(i, 0)}{\langle \delta(\alpha_0^2, 1)\rangle(1 - \langle \delta(\alpha_0^2, 1)\rangle)}. \tag{20}$$

By inversing (13), (15), (18) and (20) we easily obtain $\langle \delta(\alpha_0, 1)\rangle$, $\langle \delta(\alpha_0^2, 1)\rangle$, $\Gamma_F(i, 0)$ and $\Gamma_I(i, 0)$ as explicit functions of F_{11}, F_{12}, F_{21} and F_{22}. Also, we recall that

$$\delta(\alpha_j^2, 1) = \delta(\alpha_j, 1) + \delta(\alpha_j, -1) \qquad (\forall j). \tag{21}$$

Finally, by using translational invariance (i.e. $\langle f(\alpha_i)\rangle$ is independent of site i for an arbitrary function f, and $\langle g(\alpha_i, \alpha_j)\rangle$ only depends on the *relative* position of site i with respect to site j for an arbitrary function g) we obtain

$$m_I = \frac{F_{21}}{F_{22}} - \frac{1}{N} \tag{22}$$

$$m_F = 2\frac{F_{11}}{F_{12}} - \frac{F_{21}}{F_{22}} \tag{23}$$

$$\Gamma_I(i, j) = \frac{F_{21}}{F_{22}}(F_{22} - F_{21}) \tag{24}$$

$$\Gamma_F(i, j) = \frac{F_{11}}{F_{12}}(F_{12} - F_{11}). \tag{25}$$

To summarize let us say that we have established non-trivial relations between relevant thermal quantities and spread of damage. These relations provide an interesting manner for calculating, *at all temperatures and arbitrary lattices*, thermostatistical averages by numerically performing time averages on specific damages. The present results generalize those contained in [2] which, in turn, had generalized those in [1].

Errata of [2]: (i) Delete the number 2 in the denominator of equation (9).
(ii) Replace the number 1 by m in the denominator of equation (12).

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